THE UNIQUENESS OF THE JOSEPH IDEAL FOR THE CLASSICAL GROUPS

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ABSTRACT. The Joseph ideal is a special ideal in the universal enveloping algebra of a simple Lie algebra. For the classical groups, its uniqueness is equivalent to the existence of tensors with special properties. The existence of these tensors is usually concluded abstractly but here we present explicit formulae. This allows a rather direct computation of a parameter used in defining the Joseph ideal.

1. INTRODUCTION

Let $\mathfrak{g}$ be a simple complex Lie algebra. Let $\mathfrak{g} \odot \mathfrak{g}$ denote the Cartan product of $\mathfrak{g}$ with itself, namely the unique irreducible component of $\mathfrak{g} \otimes \mathfrak{g}$ whose highest weight is twice the highest weight of $\mathfrak{g}$ itself. Also, let us denote by

$$\mathfrak{g} \otimes \mathfrak{g} \in X \otimes Y \longmapsto X \odot Y \in \mathfrak{g} \odot \mathfrak{g} \subset \mathfrak{g} \otimes \mathfrak{g}$$

the invariant projection onto the Cartan product. Let $[X, Y]$ denote the Lie bracket and $\langle X, Y \rangle$ the Killing form on $\mathfrak{g}$. Then in the full tensor algebra $\bigotimes \mathfrak{g}$ let us consider the two-sided ideal I_λ generated by elements of the form

$$(1.1) \qquad X \otimes Y - X \odot Y - \tfrac{1}{2}[X, Y] - \lambda \langle X, Y \rangle \in \bigotimes^2 \mathfrak{g} \oplus \mathfrak{g} \oplus \mathbb{C}, \quad \forall X, Y \in \mathfrak{g}.$$

Let us denote by A_λ, the quotient algebra $\bigotimes \mathfrak{g} / I_\lambda$.

Theorem 1.1. *For each complex simple Lie algebra not isomorphic to $\mathfrak{sl}(2, \mathbb{C})$, there is precisely one value of λ for which A_λ is infinite dimensional.*

Usually, the special linear series of Lie algebras are excluded from this theorem. This is owing to the geometric realisation of certain representations usually used in defining the Joseph ideal [7]. However, as pointed out by Braverman and Joseph [2], if the result is stated as above, then the special linear algebras save for $\mathfrak{sl}(2, \mathbb{C})$ are included. The main difficulty in proving Theorem 1.1 is in showing that, for all λ save for a special value, the ideal is of finite codimension. Braverman and Joseph [2] present an abstract argument for this in general, which they make explicit for the symplectic and special linear algebras. They remark, however, that 'in general such verification seems very difficult'. This is what we accomplish for the orthogonal algebras. For completeness and convenience, we also present the special tensors that may be used in direct proofs of the symplectic and special linear cases. This covers

the classical groups. In fact, as pointed out by Gan and Savin [6], the exceptional cases are somewhat easier. These authors also tackle the classical groups by direct but still quite intricate means.

The first author would like to thank Nolan Wallach for drawing his attention to the Joseph ideal in response to a talk given at the University of California, San Diego, on higher symmetries of the Laplacian [3].

2. The orthogonal case

We shall use index conventions for tensors as is standard in differential geometry. More precisely, we use the abstract index notation of Penrose [8]. In particular g^{ab} will denote the non-degenerate quadratic form preserved by $\mathfrak{so}(n, \mathbb{C})$ and g_{ab} its inverse. We shall 'raise and lower' indices without comment: so $X^a = g^{ab}X_b$ and $X_a = g_{ab}X^b$ where a repeated index denotes the invariant pairing between vectors and covectors.

Theorem 2.1. *For $\lambda \neq -\frac{n-4}{4(n-1)(n-2)}$ and $n \geq 5$, the two-sided ideal in $\bigotimes \mathfrak{so}(n, \mathbb{C})$ generated by*

$$X \otimes Y - X \odot Y - \tfrac{1}{2}[X, Y] - \lambda \langle X, Y \rangle, \quad \text{for } X, Y \in \mathfrak{so}(n, \mathbb{C})$$

contains $\mathfrak{so}(n, \mathbb{C})$, the first graded piece of $\bigotimes \mathfrak{so}(n, \mathbb{C})$.

Proof. If we identify $\mathfrak{so}(n, \mathbb{C})$ with skew tensors in the usual fashion, then the ideal is generated by tensors of the form

$$V^{abcd} - (\odot V)^{abcd} - \tfrac{1}{2}(V^a{}_b{}^{bd} - V^d{}_b{}^{ba}) + \lambda(n-2)V^{ab}{}_{ab} \text{ for } V^{abcd} = -V^{bacd} = -V^{abdc},$$

where $(\odot V)^{abcd}$ denotes the Cartan part of V^{abcd}. (There is, of course, an explicit formula for $(\odot V)^{abcd}$ but we shall not need all of it.) Consider the following tensor

$$
\begin{aligned}
S^{abcdef} = & \ 2g^{af}g^{be}T^{cd} - 2g^{ae}g^{bf}T^{cd} - 2g^{cf}g^{de}T^{ab} + 2g^{ce}g^{df}T^{ab} \\
& + g^{ac}g^{be}T^{df} - g^{bc}g^{ae}T^{df} - g^{ad}g^{be}T^{cf} + g^{bd}g^{ae}T^{cf} \\
& - g^{ac}g^{bf}T^{de} + g^{bc}g^{af}T^{de} + g^{ad}g^{bf}T^{ce} - g^{bd}g^{af}T^{ce} \\
& - g^{ac}g^{de}T^{bf} + g^{ad}g^{ce}T^{bf} + g^{bc}g^{de}T^{af} - g^{bd}g^{ce}T^{af} \\
& + g^{ac}g^{df}T^{be} - g^{ad}g^{cf}T^{be} - g^{bc}g^{df}T^{ae} + g^{bd}g^{cf}T^{ae}.
\end{aligned}
$$

for $T^{ab} = -T^{ba}$. It is immediate that $S^{abcdef} = -S^{cdabef}$ and readily verified that

$$Z^{abcdef} \equiv \tfrac{1}{3}(S^{abcdef} + S^{abefcd}) + \tfrac{1}{6}(S^{abcedf} - S^{abdecf} - S^{abcfde} + S^{abdfce})$$

is given by

$$
\begin{aligned}
Z^{abcdef} = & \ 2g^{ce}g^{df}T^{ab} - 2g^{de}g^{cf}T^{ab} \\
& - \tfrac{1}{2}g^{ac}g^{de}T^{bf} + \tfrac{1}{2}g^{ad}g^{ce}T^{bf} + \tfrac{1}{2}g^{bc}g^{de}T^{af} - \tfrac{1}{2}g^{bd}g^{ce}T^{af} \\
& + \tfrac{1}{2}g^{ac}g^{df}T^{be} - \tfrac{1}{2}g^{ad}g^{cf}T^{be} - \tfrac{1}{2}g^{bc}g^{df}T^{ae} + \tfrac{1}{2}g^{bd}g^{cf}T^{ae} \\
& - \tfrac{1}{2}g^{ae}g^{cf}T^{bd} + \tfrac{1}{2}g^{ae}g^{df}T^{bc} + \tfrac{1}{2}g^{be}g^{cf}T^{ad} - \tfrac{1}{2}g^{be}g^{df}T^{ac} \\
& + \tfrac{1}{2}g^{af}g^{ce}T^{bd} - \tfrac{1}{2}g^{af}g^{de}T^{bc} - \tfrac{1}{2}g^{bf}g^{ce}T^{ad} + \tfrac{1}{2}g^{bf}g^{de}T^{ac}.
\end{aligned}
$$

Generally, $S^{abcdef} \mapsto Z^{abcdef}$ is the formula for the $\mathfrak{sl}(n, \mathbb{C})$-invariant projection

$$\square \otimes \square \otimes \square \longrightarrow \square \otimes \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \ .$$

In this case, however, the result is manifestly pure trace in $cdef$. It follows that under the further $\mathfrak{so}(n,\mathbb{C})$-invariant projection

$$\square \otimes \square \otimes \square \longrightarrow \square \otimes \begin{array}{c}\square\square\end{array} \longrightarrow \square \otimes \begin{array}{c}\square\square\end{array}{}_{\circ}$$

where $\circ$ denotes the trace-free part, the tensor S^{abcdef} maps to zero. But, for $n \geq 5$, this is the Cartan part in $cdef$. Its skew symmetry $S^{abcdef} = -S^{cdabef}$ ensures that the Cartan part is also zero with respect to $abcd$. Therefore, we may immediately reduce S^{abcdef} in two different ways with respect to the given ideal. We obtain, after a short calculation,

$$S^a{}_b{}^{bdef} = (n-4)[g^{af}T^{de} - g^{ae}T^{df} + g^{de}T^{af} - g^{df}T^{ae}],$$

which is skew in ad. Therefore $S^{ab}{}_{ab}{}^{cd} = 0$ and

$$S^{abcdef} \simeq \tfrac{1}{2}(S^a{}_b{}^{bdef} - S^d{}_b{}^{baef}) = (n-4)[g^{af}T^{de} - g^{ae}T^{df} + g^{de}T^{af} - g^{df}T^{ae}].$$

Tracing over de now gives

$$g^{af}T^{de} - g^{ae}T^{df} + g^{de}T^{af} - g^{df}T^{ae} \simeq (n-2)T^{af}.$$

Altogether,

(2.1) $$S^{abcdef} \simeq (n-2)(n-4)T^{af}.$$

On the other hand,

$$S^{abc}{}_d{}^{df} = -g^{af}T^{bc} + g^{bf}T^{ac} - 2(n-2)g^{cf}T^{ab} - (n-3)g^{ac}T^{bf} + (n-3)g^{bc}T^{af}$$

so

$$S^{abc}{}_d{}^{df} - S^{abf}{}_d{}^{dc} = -(n-4)[g^{ac}T^{bf} - g^{bc}T^{af} - g^{af}T^{bc} + g^{bf}T^{ac}]$$

and

$$S^{abcd}{}_{cd} = 2(n-1)(n-2)T^{ab}.$$

Therefore,

$$S^{abcdef} \simeq -\tfrac{n-4}{2}[g^{ac}T^{bf} - g^{bc}T^{af} - g^{af}T^{bc} + g^{bf}T^{ac}] - 2\lambda(n-1)(n-2)^2 T^{af}.$$

But tracing over bc gives

$$g^{ac}T^{bf} - g^{bc}T^{af} - g^{af}T^{bc} + g^{bf}T^{ac} \simeq -(n-2)T^{af}$$

and so

(2.2) $$S^{abcdef} \simeq (n-2)[\tfrac{n-4}{2} - 2\lambda(n-1)(n-2)]T^{af}.$$

Comparing (2.1) with (2.2), we conclude that T^{ab} must be in the ideal unless we have $\lambda = -\frac{n-4}{4(n-1)(n-2)}$. This is exactly what we wanted to prove. $\qquad\square$

3. The symplectic case

Theorem 3.1. *For $\lambda \neq -\frac{1}{16(n+1)}$ and $n \geq 2$, the two-sided ideal in $\bigotimes \mathfrak{sp}(2n, \mathbb{C})$ generated by*

$$X \otimes Y - X \odot Y - \tfrac{1}{2}[X, Y] - \lambda\langle X, Y \rangle, \quad \text{for } X, Y \in \mathfrak{sp}(2n, \mathbb{C})$$

contains $\mathfrak{sp}(2n, \mathbb{C})$, the first graded piece of $\bigotimes \mathfrak{sp}(2n, \mathbb{C})$.

Proof. Let ω^{ab} denote the skew form preserved by $\mathfrak{sp}(2n, \mathbb{C})$ and adopt the convention that $\omega^{ac}\omega_{bc}$ is the identity. In particular $\omega^{ab}\omega_{ab} = 2n$. If we use ω_{ab} to lower indices according to $X_b = X^a\omega_{ab}$, then we may identify $\mathfrak{sp}(2n, \mathbb{C})$ as symmetric tensors $T^{ab} = T^{ab}$ and the ideal is generated by

$$V^{abcd} - (\odot V)^{abcd} - \tfrac{1}{2}(V^a{}_b{}^{bd} + V^d{}_b{}^{ba}) + 2\lambda(n+1)V^{ab}{}_{ab} \text{ for } V^{abcd} = V^{bacd} = V^{abdc}.$$

Now consider the tensor

$$
\begin{aligned}
S^{abcdef} =\ & 4\omega^{af}\omega^{be}T^{cd} + 4\omega^{ae}\omega^{bf}T^{cd} - 4\omega^{cf}\omega^{de}T^{ab} - 4\omega^{ce}\omega^{df}T^{ab} \\
& - \omega^{ac}\omega^{be}T^{df} - \omega^{bc}\omega^{ae}T^{df} - \omega^{ad}\omega^{be}T^{cf} - \omega^{bd}\omega^{ae}T^{cf} \\
& - \omega^{ac}\omega^{bf}T^{de} - \omega^{bc}\omega^{af}T^{de} - \omega^{ad}\omega^{bf}T^{ce} - \omega^{bd}\omega^{af}T^{ce} \\
& - \omega^{ac}\omega^{de}T^{bf} - \omega^{ad}\omega^{ce}T^{bf} - \omega^{bc}\omega^{de}T^{af} - \omega^{bd}\omega^{ce}T^{af} \\
& - \omega^{ac}\omega^{df}T^{be} - \omega^{ad}\omega^{cf}T^{be} - \omega^{bc}\omega^{df}T^{ae} - \omega^{bd}\omega^{cf}T^{ae}.
\end{aligned}
$$

for $T^{ab} = T^{ba}$. It is immediate that $S^{abcdef} = -S^{cdabef}$ and readily verified that

$$Z^{abcdef} \equiv \tfrac{1}{6}(S^{abcdef} + S^{abdefd} + S^{abcfde} + S^{abfedc} + S^{abedfc} + S^{abdfec})$$

vanishes. This is already the Cartan part with respect to the $cdef$ indices. Therefore, we may reduce S^{abdcef} modulo the ideal in two different ways. We obtain

$$S^{abcdef} \simeq -2(n-1)(\omega^{ae}T^{df} + \omega^{de}T^{af} + \omega^{af}T^{de} + \omega^{df}T^{ae}) \simeq -4(n-1)(n+1)T^{af}$$

or

$$
\begin{aligned}
S^{abcdef} &\simeq -(n-1)(\omega^{ac}T^{bf} + \omega^{bc}T^{af} + \omega^{af}T^{bc} + \omega^{bf}T^{ac}) + 32\lambda(n-1)(n+1)^2 T^{af} \\
&\simeq -2(n-1)(n+1)T^{af} + 32\lambda(n-1)(n+1)^2 T^{af}.
\end{aligned}
$$

Comparing these two reductions, we see that T^{ab} lies in the ideal unless $\lambda = -\frac{1}{16(n+1)}$. This is what we wanted to prove. $\qquad\square$

Note that the critical value of λ for $\mathfrak{sp}(4, \mathbb{C}) \cong \mathfrak{so}(5, \mathbb{C})$ may be computed either from Theorem 2.1 or Theorem 3.1. Its common value is $-1/48$.

4. The special linear case

Theorem 4.1. *For $\lambda \neq -\frac{1}{8(n+1)}$ and $n \geq 3$, the two-sided ideal in $\bigotimes \mathfrak{sl}(n, \mathbb{C})$ generated by*

$$X \otimes Y - X \odot Y - \tfrac{1}{2}[X, Y] - \lambda\langle X, Y \rangle, \quad \text{for } X, Y \in \mathfrak{sl}(n, \mathbb{C})$$

contains $\mathfrak{sl}(n, \mathbb{C})$, the first graded piece of $\bigotimes \mathfrak{sl}(n, \mathbb{C})$.

Proof. If we identify $\mathfrak{sl}(n, \mathbb{C})$ with trace-free tensors $T^a{}_b$ in the usual manner, then the ideal is generated by tensors of the form

$$(4.1) \quad V^a{}_b{}^c{}_d - (\odot V)^a{}_b{}^c{}_d - \tfrac{1}{2}(V^a{}_b{}^b{}_d - V^b{}_d{}^a{}_b) - 2\lambda n V^a{}_b{}^b{}_a \quad \text{for } V^a{}_a{}^c{}_d = 0 = V^a{}_b{}^c{}_c.$$

Consider the tensor

$$\begin{aligned}
S^a{}_b{}^c{}_d{}^e{}_f &= \delta^e{}_d\delta^c{}_f T^a{}_b - \tfrac{1}{n}\delta^c{}_d\delta^e{}_f T^a{}_b - \delta^e{}_b\delta^a{}_f T^c{}_d + \tfrac{1}{n}\delta^a{}_b\delta^e{}_f T^c{}_d \\
&\quad + \delta^a{}_d\delta^e{}_b T^c{}_f - \tfrac{1}{n}\delta^a{}_d\delta^e{}_f T^c{}_b - \delta^c{}_b\delta^e{}_d T^a{}_f + \tfrac{1}{n}\delta^c{}_b\delta^e{}_f T^a{}_d
\end{aligned}$$

for $T^a{}_a = 0$. It is immediate that $S^a{}_b{}^c{}_d{}^e{}_f = -S^c{}_d{}^a{}_b{}^e{}_f$. In particular, the Cartan part of $S^a{}_b{}^c{}_d{}^e{}_f$ with respect to the indices $abcd$ vanishes. Hence, we may use (4.1) to reduce $S^a{}_b{}^c{}_d{}^e{}_f$ modulo the given ideal. We obtain

$$S^a{}_b{}^c{}_d{}^e{}_f \simeq -(n-1)\delta^e{}_d T^a{}_f - \delta^a{}_f T^e{}_d + \delta^a{}_d T^e{}_f + \delta^e{}_f T^a{}_d \simeq -\tfrac{1}{2}n(n-2)T^a{}_f.$$

On the other hand, it is readily verified that

$$Z^a{}_b{}^c{}_d{}^e{}_f \equiv \tfrac{1}{4}(S^a{}_b{}^c{}_d{}^e{}_f + S^a{}_b{}^e{}_d{}^c{}_f + S^a{}_b{}^c{}_f{}^e{}_d + S^a{}_b{}^e{}_f{}^c{}_d)$$

is given by

$$\begin{aligned}
Z^a{}_b{}^c{}_d{}^e{}_f &= \tfrac{1}{2}\delta^e{}_d\delta^c{}_f T^a{}_b - \tfrac{1}{4}\delta^c{}_b\delta^e{}_d T^a{}_f - \tfrac{1}{2n}\delta^c{}_d\delta^e{}_f T^a{}_b + \tfrac{1}{4n}\delta^a{}_b\delta^e{}_f T^c{}_d \\
&\quad - \tfrac{1}{4n}\delta^a{}_d\delta^e{}_f T^c{}_b + \tfrac{1}{4n}\delta^c{}_b\delta^e{}_f T^a{}_d - \tfrac{1}{4}\delta^e{}_b\delta^c{}_f T^a{}_d - \tfrac{1}{4n}\delta^c{}_d\delta^a{}_f T^e{}_b \\
&\quad + \tfrac{1}{4n}\delta^a{}_b\delta^e{}_d T^c{}_f + \tfrac{1}{4n}\delta^a{}_b\delta^c{}_f T^e{}_d + \tfrac{1}{4n}\delta^c{}_d\delta^e{}_b T^a{}_f - \tfrac{1}{2n}\delta^c{}_f\delta^e{}_d T^a{}_b \\
&\quad + \tfrac{1}{4n}\delta^a{}_b\delta^c{}_d T^e{}_f - \tfrac{1}{4n}\delta^a{}_f\delta^e{}_d T^c{}_b + \tfrac{1}{4n}\delta^c{}_b\delta^e{}_d T^a{}_f - \tfrac{1}{4n}\delta^a{}_d\delta^c{}_f T^e{}_b \\
&\quad + \tfrac{1}{4n}\delta^e{}_b\delta^c{}_f T^a{}_d + \tfrac{1}{2}\delta^e{}_f\delta^c{}_d T^a{}_b - \tfrac{1}{4}\delta^c{}_b\delta^e{}_f T^a{}_d - \tfrac{1}{4}\delta^e{}_b\delta^c{}_d T^a{}_f.
\end{aligned}$$

Generally, $S^a{}_b{}^c{}_d{}^e{}_f \mapsto Z^a{}_b{}^c{}_d{}^e{}_f$ followed by the removal of all traces in the $cdef$ indices is the Cartan projection in these indices. In this case, however, $Z^a{}_b{}^c{}_d{}^e{}_f$ is manifestly pure trace and so this Cartan part of $S^a{}_b{}^c{}_d{}^e{}_f$ vanishes. This allows us to use (4.1) with respect to the $cdef$ indices to conclude that

$$\begin{aligned}
S^a{}_b{}^c{}_d{}^e{}_f &\simeq \tfrac{1}{2}(\delta^c{}_f T^a{}_b - (n-1)\delta^c{}_b T^a{}_f + \delta^a{}_b T^c{}_f - \delta^a{}_f T^c{}_b) + 2\lambda n(n-2)(n+1)T^a{}_f \\
&\simeq -\tfrac{1}{4}n(n-2)T^a{}_f + 2\lambda n(n-2)(n+1)T^a{}_f.
\end{aligned}$$

Comparing this with our previous reduction, we see that $T^a{}_b$ lies in the ideal unless $\lambda = -\frac{1}{8(n+1)}$. This is what we wanted to prove. $\qquad\square$

Note that the critical value of λ for $\mathfrak{sl}(4, \mathbb{C}) \cong \mathfrak{so}(6, \mathbb{C})$ may be computed either from Theorem 2.1 or Theorem 4.1. Its common value is $-1/40$.

5. REMARKS AND CONCLUSIONS

We should explain how the special tensors used in the proofs of Theorems 2.1, 3.1, and 4.1 arise. For all simple Lie algebras other than the special linear series, there is a common source as follows. Let $\mathfrak{g}$ denote a complex simple Lie algebra and let Φ denote the composition

$$\Lambda^2\mathfrak{g} \otimes \mathfrak{g} \hookrightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\text{Id} \otimes\_\odot\_} \mathfrak{g} \otimes \odot^2\mathfrak{g}.$$

Theorem 5.1. *For any simple complex Lie algebra* $\mathfrak{g}$ *not isomorphic to* $\mathfrak{sl}(n,\mathbb{C})$,

$$\dim \operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \Lambda^2\mathfrak{g} \otimes \mathfrak{g}) = 2 \quad and \quad \dim \operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \mathfrak{g} \otimes \odot^2\mathfrak{g}) = 1.$$

Proof. A case-by-case verification using, for example, Klimyk's formula.   □

Corollary 5.2. *For any simple complex Lie algebra* $\mathfrak{g}$ *not isomorphic to* $\mathfrak{sl}(n,\mathbb{C})$,

$$\dim \operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \ker \Phi) \geq 1.$$

This result is used abstractly by Braverman and Joseph [2] and our proofs are very much motivated by this approach: in proving Theorems 2.1 and 3.1 we find explicit non-zero homomorphisms $\mathfrak{g} \to \ker \Phi$. In fact, it is easily verified that $\operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \ker \Phi)$ is 1-dimensional so our homomorphisms are unique up to scale.

For the special linear algebras the dimensions are different:–

Theorem 5.3. *For* $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$,

$$\dim \operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \Lambda^2\mathfrak{g} \otimes \mathfrak{g}) = \begin{cases} 4 & \text{if } n \geq 3 \\ 1 & \text{if } n = 2 \end{cases} \quad and \quad \dim \operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \mathfrak{g} \otimes \odot^2\mathfrak{g}) = 1, \ \forall n \geq 2.$$

Corollary 5.4. *For* $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ *with* $n \geq 3$,

$$\dim \operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \ker \Phi) \geq 3.$$

In fact, using tensors, we have checked that $\operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \ker \Phi)$ is 3-dimensional and within it there is a 2-dimensional subspace $\operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \ker \Phi \cap \ker \Psi)$ where Ψ is the composition

$$\Lambda^2\mathfrak{g} \otimes \mathfrak{g} \hookrightarrow \mathfrak{g} \otimes \mathfrak{g} \otimes \mathfrak{g} \xrightarrow{\operatorname{Id} \otimes \langle\_,\_\rangle} \mathfrak{g} \otimes \mathbb{C} = \mathfrak{g}.$$

Any homomorphism in $\operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \ker \Phi) \backslash \operatorname{Hom}_\mathfrak{g}(\mathfrak{g}, \ker \Phi \cap \ker \Psi)$ will suffice for deriving the critical value of λ as in our proof of Theorem 4.1. This critical value of λ is also obtained by Braverman and Joseph [2, §7.4 and §7.7]. They also remark [2, §5.4] that the symplectic case may be dealt with by an 'extremely rare' but 'simple-minded procedure' going back to Dirac. From the tensorial point of view, the reason for this is that if one naïvely extends a tensor

$$S \in \left(\Lambda^2\mathfrak{sp}(2m,\mathbb{C}) \otimes \mathfrak{sp}(2m,\mathbb{C})\right) \cap \left(\mathfrak{sp}(2m,\mathbb{C}) \otimes \odot^2\mathfrak{sp}(2m,\mathbb{C})\right)$$

by adding zero components then one obtains a tensor in

$$\left(\Lambda^2\mathfrak{sp}(2n,\mathbb{C}) \otimes \mathfrak{sp}(2n,\mathbb{C})\right) \cap \left(\mathfrak{sp}(2n,\mathbb{C}) \otimes \odot^2\mathfrak{sp}(2n,\mathbb{C})\right)$$

for any $n > m$. In effect, Braverman and Joseph use this observation and an explicit tensor for the case $n = 2$ to obtain the general case.

Usually, Theorem 1.1 is stated in terms of the universal enveloping algebra $\mathfrak{U}(\mathfrak{g})$ of $\mathfrak{g}$. To do this, notice that the generators (1.1) of I_λ may be split into skew and symmetric parts:–

$$X \otimes Y - Y \otimes X - [X,Y] \quad \text{and} \quad X \otimes Y + Y \otimes X - 2X \odot Y - 2\lambda\langle X,Y\rangle$$

and that we may define the algebra A_λ in two steps, firstly taking the quotient of the tensor algebra by the skew generators. This gives $\mathfrak{U}(\mathfrak{g})$ and an image ideal $\bar{I}_\lambda$ so that $A_\lambda = \mathfrak{U}(\mathfrak{g})/\bar{I}_\lambda$. What we have shown more precisely in §2–§4 is the following:–

Theorem 5.5. *For the classical complex simple algebras*

$$\mathfrak{g} = \mathfrak{so}(n, \mathbb{C}), \quad n \geq 5, \quad \lambda \neq -\frac{n-4}{4(n-1)(n-2)}$$
$$\mathfrak{g} = \mathfrak{sp}(2n, \mathbb{C}), \quad n \geq 2, \quad \lambda \neq -\frac{1}{16(n+1)}$$
$$\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C}), \quad n \geq 3, \quad \lambda \neq -\frac{1}{8(n+1)}$$

the ideal $\bar{I}_\lambda$ coincides with $\mathfrak{U}(\mathfrak{g})$ if $\lambda \neq 0$ whilst $\bar{I}_0 = \mathfrak{U}_+(\mathfrak{g}) \subset \mathfrak{U}(\mathfrak{g})$, the unique maximal ideal consisting of elements without constant part.

Proof. Theorems 2.1, 3.1, and 4.1 say that, in these circumstances, the ideal I_λ contains $\mathfrak{g}$ and hence contains $\bigoplus_{s \geq 1} \bigotimes^s \mathfrak{g}$, whose image is $\mathfrak{U}_+(\mathfrak{g})$ by definition. The conclusions are now immediate from (1.1). $\qquad\square$

In all other cases the algebra A_λ is, in fact, infinite-dimensional. For the orthogonal algebras, for example, there are linear differential operators

$$\mathcal{D}_X \quad \text{for all } X \in \bigodot^s \mathfrak{so}(m+1, 1)$$

constructed in [3] that satisfy

$$\mathcal{D}_X \mathcal{D}_Y = \mathcal{D}_{X \odot Y} + \tfrac{1}{2}\mathcal{D}_{[X,Y]} - \frac{m-2}{4m(m+1)}\mathcal{D}_{\langle X, Y \rangle}, \quad \forall X, Y \in \mathfrak{so}(m+1, 1).$$

The corresponding holomorphic differential operators provide a realisation of A_λ for $\mathfrak{g} = \mathfrak{so}(n, \mathbb{C})$ and $\lambda = -\frac{(n-4)}{4(n-1)(n-2)}$. There are similar linear holomorphic differential operators for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ constructed as follows. Recall that in §4 we identified $\mathfrak{sl}(n, \mathbb{C})$ with trace-free tensors $X^a{}_b$. More generally,

$$\bigodot^s \mathfrak{sl}(n, \mathbb{C}) = \left\{ X\underbrace{{}^a{}_b{}^c{}_d{}^{\cdots}{}_{\cdots}{}^e{}_f}_{2s \text{ indices}} \quad \text{s.t.} \quad \begin{array}{l} X^a{}_b{}^c{}_d{}^{\cdots}{}_{\cdots}{}^e{}_f = X^{(a}{}_{(b}{}^c{}_d{}^{\cdots}{}_{\cdots}{}^{e)}{}_{f)} \\ X^a{}_b{}^c{}_d{}^{\cdots}{}_{\cdots}{}^e{}_f \text{ is totally trace-free.} \end{array} \right\}$$

and we define

$$\mathcal{D}_X \equiv (-1)^s X^a{}_b{}^c{}_d{}^{\cdots}{}_{\cdots}{}^e{}_f Z^b Z^d \cdots Z^f \frac{\partial^s}{\partial Z^a \partial Z^c \cdots \partial Z^e}$$

as a holomorphic differential operator acting on $\mathbb{C}^n$. For $X^a{}_b, Y^c{}_d \in \mathfrak{sl}(n, \mathbb{C})$,

$$\mathcal{D}_X \mathcal{D}_Y - \mathcal{D}_Y \mathcal{D}_X = (Y^a{}_c X^c{}_b - X^a{}_c Y^c{}_b) Z^b \frac{\partial}{\partial Z^a} = -[X, Y]^a{}_b Z^b \frac{\partial}{\partial Z^a} = \mathcal{D}_{[X,Y]}$$

and

$$\mathcal{D}_X \mathcal{D}_Y + \mathcal{D}_Y \mathcal{D}_X = 2 X^{(a}{}_{(b} Y^{c)}{}_{d)} Z^b Z^d \frac{\partial^2}{\partial Z^a \partial Z^c} + (X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a}.$$

However, if we write

$$X^{(a}{}_{(b} Y^{c)}{}_{d)} = C^{ac}{}_{bd} + D^{(a}{}_{(b} \delta^{c)}{}_{d)} + E \delta^{(a}{}_{(b} \delta^{c)}{}_{d)},$$

where

$$D^a{}_b = \frac{1}{n+2}(X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) - \frac{2}{n(n+2)} X^c{}_d Y^d{}_c \delta^a{}_b \quad \text{and} \quad E = \frac{1}{n(n+1)} X^c{}_d Y^d{}_c,$$

then $C^{ac}{}_{bd}$ and $D^a{}_b$ are trace-free. In particular, $C^{ac}{}_{bd} = (X \odot Y)^{ac}{}_{bd}$ and

$$
\begin{aligned}
\mathcal{D}_X \mathcal{D}_Y + \mathcal{D}_Y \mathcal{D}_X &= 2(C^{ac}{}_{bd} + D^c{}_b \delta^a{}_d + E\delta^c{}_b \delta^a{}_d) Z^b Z^d \frac{\partial^2}{\partial Z^a \partial Z^c} \\
&\quad + (X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a} \\
&= 2\mathcal{D}_{X \odot Y} + 2D^a{}_b Z^b Z^c \frac{\partial^2}{\partial Z^c \partial Z^a} + 2E Z^b Z^c \frac{\partial^2}{\partial Z^c \partial Z^b} \\
&\quad + (X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a}.
\end{aligned}
$$

Now, let us restrict the action of these differential operators to germs ϕ of holomorphic functions defined near some basepoint in $\mathbb{C}^n \setminus \{0\}$ and 'homogeneous of degree w' in the sense that $Z^a \partial/\partial Z^a \phi = w\phi$. We find that

$$
\begin{aligned}
\mathcal{D}_X \mathcal{D}_Y + \mathcal{D}_Y \mathcal{D}_X &= 2\mathcal{D}_{X \odot Y} + (2\tfrac{w-1}{n+2} + 1)(X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a} \\
&\quad + 2(\tfrac{1}{n(n+1)} - \tfrac{2}{n(n+2)}) w(w-1) X^c{}_d Y^d{}_c \\
&= 2\mathcal{D}_{X \odot Y} + \tfrac{2w+n}{n+2}(X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a} \\
&\quad - 2\tfrac{w(w-1)}{(n+1)(n+2)} X^c{}_d Y^d{}_c.
\end{aligned}
$$

Assembling these computations we conclude that

$$
\mathcal{D}_X \mathcal{D}_Y = \mathcal{D}_{X \odot Y} + \tfrac{2w+n}{2(n+2)}(X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b) Z^b \frac{\partial}{\partial Z^a} + \tfrac{1}{2}\mathcal{D}_{[X,Y]} - \tfrac{w(w-1)}{2n(n+1)(n+2)} \mathcal{D}_{\langle X,Y \rangle}.
$$

In particular, for $w = -n/2$ we obtain

$$
\mathcal{D}_X \mathcal{D}_Y = \mathcal{D}_{X \odot Y} + \tfrac{1}{2}\mathcal{D}_{[X,Y]} - \tfrac{1}{8(n+1)} \mathcal{D}_{\langle X,Y \rangle}.
$$

These operators provide a realisation of A_λ for $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $\lambda = -\tfrac{1}{8(n+1)}$ and, in particular, show that this algebra is infinite dimensional.

If $n = 2$ we can proceed further because, in this case,

$$
X^a{}_c Y^c{}_b + Y^a{}_c X^c{}_b = X^d{}_c Y^c{}_d \delta^a{}_b = \tfrac{1}{4}\langle X, Y \rangle \delta^a{}_b
$$

whence

$$
\mathcal{D}_X \mathcal{D}_Y = \mathcal{D}_{X \odot Y} + \tfrac{1}{2}\mathcal{D}_{[X,Y]} + \tfrac{w(w+2)}{24} \mathcal{D}_{\langle X,Y \rangle}
$$

for any $w \in \mathbb{C}$. In particular, this shows that A_λ is infinite-dimensional for $\mathfrak{sl}(2, \mathbb{C})$ no matter what is λ.

An alternative to these geometric realisations of A_λ is provided by the generalised Poincaré-Birkhoff-Witt Theorem of Braverman and Gaitsgory [1], which enables one to identify the associated graded algebra $\mathrm{gr}(A_\lambda)$. Specifically, if we let $R \subset \mathfrak{g} \otimes \mathfrak{g}$ be the $\mathfrak{g}$-invariant complement to $\mathfrak{g} \odot \mathfrak{g}$ and $J(R)$ be the two-sided ideal in $\bigotimes \mathfrak{g}$ generated by R, then as a special case of [1] we obtain criteria under which the canonical surjection $p : \bigotimes \mathfrak{g}/J(R) \to \mathrm{gr}(A_\lambda)$ of graded algebras is an isomorphism. These criteria are then verified by Braverman and Joseph [2] in the case of critical λ. It follows from a result of Kostant (given in a lecture at MIT in 1980 and explained with proof in [5, Chapter 3]) that the graded algebra $\bigotimes \mathfrak{g}/J(R)$ is simply the Cartan

algebra $\bigodot \mathfrak{g} = \bigoplus_{s=0}^{\infty} \bigodot^s \mathfrak{g}$ for any complex simple Lie algebra $\mathfrak{g}$. This is also proved by tensorial means in [3] for the orthogonal algebras, in [4] for the special linear algebras, and the symplectic algebras are easily dealt with by a similar argument. In [4], however, it was incorrectly asserted that p is always an isomorphism.

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